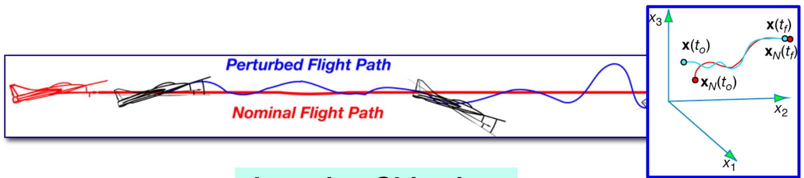
Linearized Equations of Motion

Robert Stengel, Aircraft Flight Dynamics
MAE 331, 2018



Learning Objectives

Develop linear equations to describe small perturbational motions

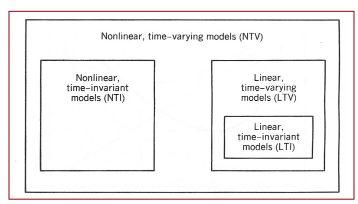
Apply to aircraft dynamic equations

Reading:
Flight Dynamics
234-242, 255-266, 274-297, 321-325, 329-330

http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html

1

How Is System Response Calculated?



- **Linear and nonlinear, time-varying and time-invariant dynamic models**
  - Numerical integration ("time domain")
- **Linear, time-invariant (LTI) dynamic models**
  - Numerical integration ("time domain")
  - State transition ("time domain")
  - Transfer functions ("frequency domain")

2

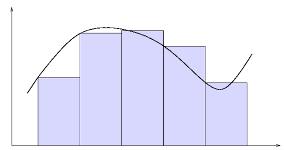
Integration Algorithms

- **Exact**

$$\mathbf{x}(T) = \mathbf{x}(0) + \int_0^T \mathbf{f}\left[\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t)\right] dt$$

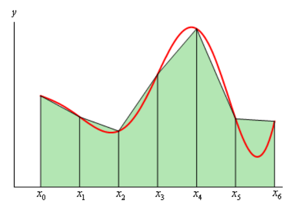
- **Rectangular (Euler) Integration**

$$\mathbf{x}(t_k) = \mathbf{x}(t_{k-1}) + \delta\mathbf{x}(t_{k-1}, t_k)$$
$$\approx \mathbf{x}(t_{k-1}) + \mathbf{f}\left[\mathbf{x}(t_{k-1}), \mathbf{u}(t_{k-1}), \mathbf{w}(t_{k-1})\right]\delta t$$
$$\delta t = t_k - t_{k-1}$$



- **Trapezoidal (modified Euler) Integration** *(~MATLAB's ode23)*

$$\mathbf{x}(t_k) \approx \mathbf{x}(t_{k-1}) + \frac{1}{2}\left[\delta\mathbf{x}_1 + \delta\mathbf{x}_2\right]$$
where
$$\delta\mathbf{x}_1 = \mathbf{f}\left[\mathbf{x}(t_{k-1}), \mathbf{u}(t_{k-1}), \mathbf{w}(t_{k-1})\right]\delta t$$
$$\delta\mathbf{x}_2 = \mathbf{f}\left[\mathbf{x}(t_{k-1}) + \delta\mathbf{x}_1, \mathbf{u}(t_k), \mathbf{w}(t_k)\right]\delta t$$



See MATLAB manual for descriptions of *ode45* and *ode15s*

3

Numerical Integration: MATLAB Ordinary Differential Equation Solvers*

- **Explicit Runge-Kutta Algorithm**
- **Adams-Bashforth-Moulton Algorithm**
- **Numerical Differentiation Formula**
- **Modified Rosenbrock Method**
- **Trapezoidal Rule**
- **Trapezoidal Rule w/Back Differentiation**

Solver	Problem Type	Order of Accuracy	When to Use
ode45	Nonstiff	Medium	Most of the time. This should be the first solver you try.
ode23	Nonstiff	Low	For problems with crude error tolerances or for solving moderately stiff problems.
ode113	Nonstiff	Low to high	For problems with stringent error tolerances or for solving computationally intensive problems.
ode15s	Stiff	Low to medium	If ode45 is slow because the problem is stiff.
ode23s	Stiff	Low	If using crude error tolerances to solve stiff systems and the mass matrix is constant.
ode23t	Moderately Stiff	Low	For moderately stiff problems if you need a solution without numerical damping.
ode23tb	Stiff	Low	If using crude error tolerances to solve stiff systems.

* http://www.mathworks.com/access/helpdesk/help/techdoc/index.html?/access/helpdesk/help/techdoc/ref/ode23.html.
Shampine, L. F. and M. W. Reichelt, "The MATLAB ODE Suite," *SIAM Journal on Scientific Computing*, Vol. 18, 1997, pp 1-22.

4

Nominal and Actual Trajectories

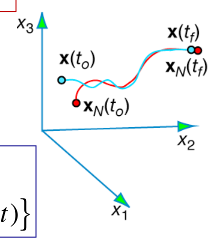
- **Nominal (or reference) trajectory and control history**

$$\left\{\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t)\right\} \quad \text{for } t \text{ in } [t_o, t_f]$$

$\mathbf{x}$: *dynamic state*	
$\mathbf{u}$: *control input*	
$\mathbf{w}$: *disturbance input*	

$$\mathbf{x}_N(T) = \mathbf{x}_N(0) + \int_0^T \mathbf{f}\left[\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t)\right] dt$$

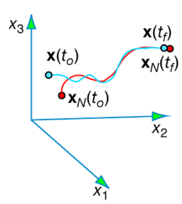
- **Actual trajectory perturbed by**
  - Small initial condition variation, $\Delta\mathbf{x}_o(t_o)$
  - Small control variation, $\Delta\mathbf{u}(t)$



$$\left\{\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t)\right\} \quad \text{for } t \text{ in } [t_o, t_f]$$
$$= \left\{\mathbf{x}_N(t) + \Delta\mathbf{x}(t), \mathbf{u}_N(t) + \Delta\mathbf{u}(t), \mathbf{w}_N(t) + \Delta\mathbf{w}(t)\right\}$$

$$\mathbf{x}(T) = \mathbf{x}(0) + \int_0^T \mathbf{f}\left[\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t)\right] dt$$

5



Both Paths Satisfy the Dynamic Equations

Dynamic models for the actual and the nominal problems are the same

$$\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t)], \quad \mathbf{x}_N(t_o) \ \text{given}$$
$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t)], \quad \mathbf{x}(t_o) \ \text{given}$$

Differences in initial condition and forcing ...

$$\Delta\mathbf{x}(t_o) = \mathbf{x}(t_o) - \mathbf{x}_N(t_o)$$
$$\left\{\begin{array}{l} \Delta\mathbf{u}(t) = \mathbf{u}(t) - \mathbf{u}_N(t) \\ \Delta\mathbf{w}(t) = \mathbf{w}(t) - \mathbf{w}_N(t) \end{array}\right\} \text{ in } \left[t_o, t_f\right]$$

... perturb rate of change and the state

$$\left\{\begin{array}{l} \dot{\mathbf{x}}(t) = \dot{\mathbf{x}}_N(t) + \Delta\dot{\mathbf{x}}(t) \\ \mathbf{x}(t) = \mathbf{x}_N(t) + \Delta\mathbf{x}(t) \end{array}\right\} \text{ in } \left[t_o, t_f\right]$$

6

Approximate Neighboring Trajectory as Linear Perturbation to Nominal Trajectory

New trajectory [*change in notation*]

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}_N(t) + \Delta\dot{\mathbf{x}}(t)$$
$$= \mathbf{f}[\mathbf{x}_N(t) + \Delta\mathbf{x}(t), \mathbf{u}_N(t) + \Delta\mathbf{u}(t), \mathbf{w}_N(t) + \Delta\mathbf{w}(t), t]$$

New trajectory ~ nominal path + linear perturbation

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}_N(t) + \Delta\dot{\mathbf{x}}(t)$$
$$\approx \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t), t] + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\Delta\mathbf{x}(t) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\Delta\mathbf{u}(t) + \frac{\partial \mathbf{f}}{\partial \mathbf{w}}\Delta\mathbf{w}(t)$$

7

Perturbation Dynamics Approximated by Linearized Equation

- **Solve for the nominal and perturbation trajectories <u>separately</u>**

Nominal Equation

$$\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t), t], \quad \mathbf{x}_N(t_o) \text{ given}$$

Perturbation Equation

$$\Delta\dot{\mathbf{x}}(t) \approx \left[\left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} \Delta\mathbf{x}(t) \right] + \left[\left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} \Delta\mathbf{u}(t) \right] + \left[\left.\frac{\partial \mathbf{f}}{\partial \mathbf{w}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} \Delta\mathbf{w}(t) \right]$$
$$\triangleq \mathbf{F}(t)\Delta\mathbf{x}(t) + \mathbf{G}(t)\Delta\mathbf{u}(t) + \mathbf{L}(t)\Delta\mathbf{w}(t), \quad \Delta\mathbf{x}(t_o) \text{ given}$$

$$\dim(\mathbf{x}) = \dim(\Delta\mathbf{x}) = n \times 1$$
$$\dim(\mathbf{u}) = \dim(\Delta\mathbf{u}) = m \times 1$$
$$\dim(\mathbf{w}) = \dim(\Delta\mathbf{w}) = s \times 1$$

8

Jacobian Matrices Express Solution Sensitivity to Small Perturbations

Stability matrix, F, is square

$$\mathbf{F}(t)=\left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}=\left[\begin{array}{ccc}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \cdots & \cdots & \cdots\\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}\end{array}\right]_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}$$

$\dim(\mathbf{F}) = n \times n$

9

Sensitivity to Control Perturbations, G

$$\mathbf{G}(t)=\left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}=\left[\begin{array}{ccc}\frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m}\\ \cdots & \cdots & \cdots\\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m}\end{array}\right]_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}$$

$\dim(\mathbf{G}) = n \times m$

10

Sensitivity to Disturbance Perturbations, G

$$\mathbf{L}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{w}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}} = \begin{bmatrix} \frac{\partial f_1}{\partial w_1} & \cdots & \frac{\partial f_1}{\partial w_s} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial w_1} & \cdots & \frac{\partial f_n}{\partial w_s} \end{bmatrix}_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}$$

$\dim(\mathbf{L}) = n \times s$

11

Scalar Example of Nominal and Perturbation Equations

Actual Nonlinear System

$$\dot{x}(t) = x(t) + 2x^2(t) + 3u(t) + 4w^3(t)$$

Nominal Nonlinear System

$$\dot{x}_N(t) = x_N(t) + 2x_N{}^2(t) + 3u_N(t) + 4w_N{}^3(t)$$

Linear Perturbation System

$$\Delta\dot{x}(t) = \Delta x(t) + 4x_N(t)\Delta x(t) + 3\Delta u(t) + 12w_N^2(t)\Delta w(t)$$

Time-Varying Parameter

Time-Varying Parameter

12

Comparison of Damped Linear and Nonlinear Systems

Linear Spring

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -10x_1(t) - x_2(t)$$

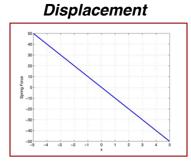
Displacement

Rate of Change

Spring

Damper

Linear Spring Force vs. Displacement



Linear plus Stiffening Cubic Spring

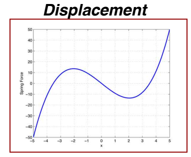
$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -10x_1(t) - 10x_1^3(t) - x_2(t)$$

Cubic Spring Force vs. Displacement

Linear plus Weakening Cubic Spring

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -10x_1(t) + 0.8x_1^3(t) - x_2(t)$$

Cubic Spring Force vs. Displacement



13

MATLAB Simulation of Linear and Nonlinear Dynamic Systems

MATLAB Main Script

```
% Nonlinear and Linear Examples
clear
tspan = [0 10];
xo = [0, 10];
[t1,x1 = ode23('NonLin',tspan,xo);
xo = [0, 1];
[t2,x2] = ode23('NonLin',tspan,xo);
xo = [0, 10];
[t3,x3] = ode23('Lin',tspan,xo);
xo = [0, 1];
[t4,x4] = ode23('Lin',tspan,xo);

subplot(2,1,1)
plot(t1,x1(:,1),'k',t2,x2(:,1),'b',t3,x3(:,1),'r',t4,x4(:,1),'g')
ylabel('Position'), grid
subplot(2,1,2)
plot(t1,x1(:,2),'k',t2,x2(:,2),'b',t3,x3(:,2),'r',t4,x4(:,2),'g')
xlabel('Time'), ylabel('Rate'), grid
```

Linear System

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -10x_1(t) - x_2(t)$$

```
function xdot = Lin(t,x)
% Linear Ordinary Differential Equation
% x(1) = Position
% x(2) = Rate
xdot = [x(2)
    -10*x(1) - x(2)];
```

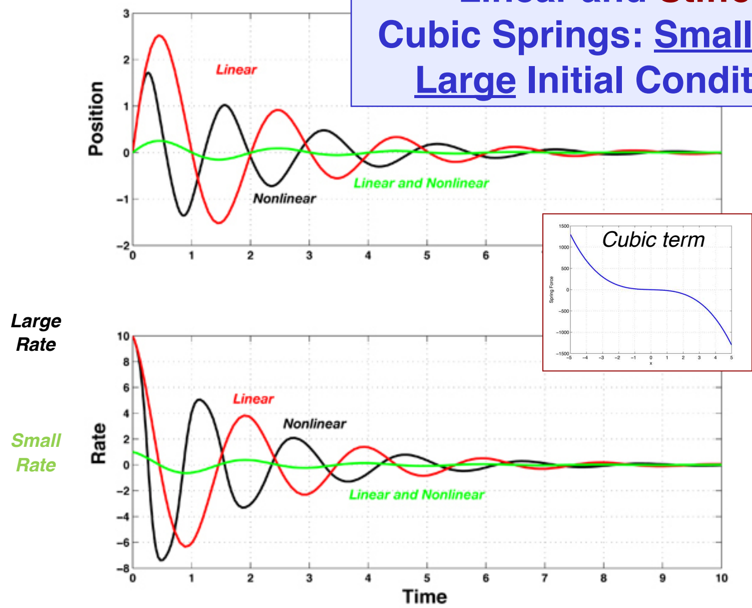
Nonlinear System

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -10x_1(t) + 0.8x_1^3(t) - x_2(t)$$

```
function xdot = NonLin(t,x)
% Nonlinear Ordinary Differential Equation
% x(1) = Position
% x(2) = Rate
xdot = [x(2)
    -10*x(1) + 0.8*x(1)^3 - x(2)];
```

14

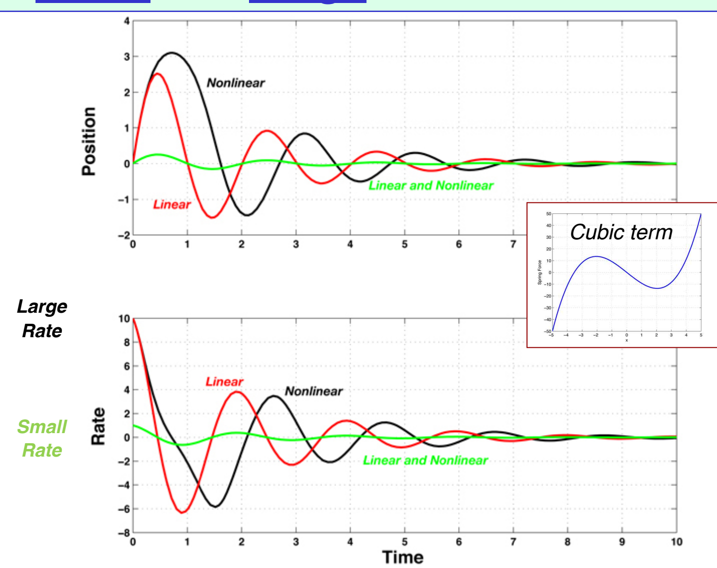
Linear and Stiffening Cubic Springs: Small and Large Initial Conditions



Linear and nonlinear responses are indistinguishable with small initial condition

15

Linear and Weakening Cubic Springs: Small and Large Initial Conditions

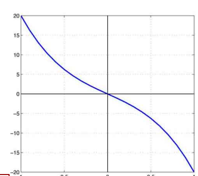


16

Linear, Time-Varying (LTV) Approximation of Perturbation Dynamics

17

Stiffening Linear-Cubic Spring Example



Nonlinear, time-invariant (NTI) equation

$$\dot{x}_1(t) = f_1 = x_2(t)$$
$$\dot{x}_2(t) = f_2 = -10x_1(t) - 10x_1^3(t) - x_2(t)$$

Integrate nonlinear equations to produce nominal path

$$\begin{bmatrix} x_{1_N}(0) \\ x_{2_N}(0) \end{bmatrix} \Rightarrow \int_0^{t_f} \begin{bmatrix} f_{1_N} \\ f_{2_N} \end{bmatrix} dt \Rightarrow \begin{bmatrix} x_{1_N}(t) \\ x_{2_N}(t) \end{bmatrix} \quad in \quad [0, t_f]$$

Analytical evaluation of partial derivatives

$$\frac{\partial f_1}{\partial x_1} = 0; \qquad \frac{\partial f_1}{\partial x_2} = 1$$
$$\frac{\partial f_2}{\partial x_1} = -10 - 30x_{1_N}^2(t); \qquad \frac{\partial f_2}{\partial x_2} = -1$$

$$\frac{\partial f_1}{\partial u} = 0; \quad \frac{\partial f_1}{\partial w} = 0$$
$$\frac{\partial f_2}{\partial u} = 0; \quad \frac{\partial f_2}{\partial w} = 0$$

18

Nominal (NTI) and Perturbation (LTV) Dynamic Equations

Nonlinear, time-invariant (NTI) nominal equation

$$\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t)], \quad \mathbf{x}_N(0)\ given$$

$$\dot{x}_{1_N}(t) = x_{2_N}(t)$$

$$\dot{x}_{2_N}(t) = -10x_{1_N}(t) - 10x_{1_N}^{\;3}(t) - x_{2_N}(t)$$

Example

$$\begin{bmatrix} x_{1_N}(0) \\ x_{2_N}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

Perturbations approximated by linear, time-varying (LTV) equation

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}(t)\Delta\mathbf{x}(t), \quad \Delta\mathbf{x}(0)\ given$$

$$\begin{bmatrix} \Delta\dot{x}_1(t) \\ \Delta\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\left(10 + 30x_{1_N}^{\;2}(t)\right) & -1 \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}$$

Example

$$\begin{bmatrix} \Delta x_1(0) \\ \Delta x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

19

Comparison of Approximate and Exact Solutions

$$x_{2_N}(0) = 9$$

$$\Delta x_2(0) = 1$$

$$x_{2_N}(t) + \Delta x_2(t) = 10$$

$$x_2(t) = 10$$

$\mathbf{x}_N(t)$

$\Delta\mathbf{x}(t)$

$\mathbf{x}_N(t) + \Delta\mathbf{x}(t)$

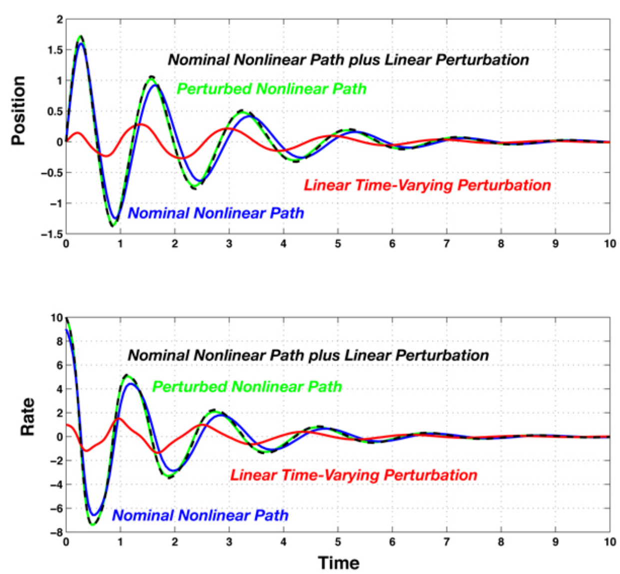
$\mathbf{x}(t)$

$\dot{\mathbf{x}}_N(t)$

$\Delta\dot{\mathbf{x}}(t)$

$\dot{\mathbf{x}}_N(t) + \Delta\dot{\mathbf{x}}(t)$

$\dot{\mathbf{x}}(t)$



20

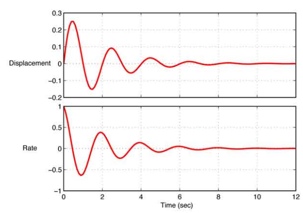
Suppose Nominal Initial Condition is Zero

Nominal solution remains at equilibrium

$$\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t)], \quad \mathbf{x}_N(0) = 0, \quad \mathbf{x}_N(t) = 0 \text{ in } \left[0,\infty\right]$$

Perturbation equation is linear and time-invariant (LTI)

$$\begin{bmatrix} \Delta\dot{x}_1(t) \\ \Delta\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \left[-10-\cancel{30(0)}\right] & -1 \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}$$



21

Separation of the Equations of Motion into Longitudinal and Lateral-Directional Sets

22

Rigid-Body Equations of Motion

Rate of change of Translational Velocity

$$\dot{u} = X/m - g\sin\theta + rv - qw$$
$$\dot{v} = Y/m + g\sin\phi\cos\theta - ru + pw$$
$$\dot{w} = Z/m + g\cos\phi\cos\theta + qu - pv$$

Rate of change of Translational Position

$$\dot{x}_I = (\cos\theta\cos\psi)u + (-\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi)v + (\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi)w$$
$$\dot{y}_I = (\cos\theta\sin\psi)u + (\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi)v + (-\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi)w$$
$$\dot{z}_I = (-\sin\theta)u + (\sin\phi\cos\theta)v + (\cos\phi\cos\theta)w$$

Rate of change of Angular Velocity

($I_{xy} = I_{yz} = 0$)

$$\dot{p} = \left(I_{zz}L + I_{xz}N - \left\{I_{xz}\left(I_{yy} - I_{xx} - I_{zz}\right)p + \left[I_{xz}^2 + I_{zz}\left(I_{zz} - I_{yy}\right)\right]r\right\}q\right) \div \left(I_{xx}I_{zz} - I_{xz}^2\right)$$
$$\dot{q} = \left[M - \left(I_{xx} - I_{zz}\right)pr - I_{xz}\left(p^2 - r^2\right)\right] \div I_{yy}$$
$$\dot{r} = \left(I_{xz}L + I_{xx}N - \left\{I_{xz}\left(I_{yy} - I_{xx} - I_{zz}\right)r + \left[I_{xz}^2 + I_{xx}\left(I_{xx} - I_{yy}\right)\right]p\right\}q\right) \div \left(I_{xx}I_{zz} - I_{xz}^2\right)$$

Rate of change of Angular Position

$$\dot{\phi} = p + (q\sin\phi + r\cos\phi)\tan\theta$$
$$\dot{\theta} = q\cos\phi - r\sin\phi$$
$$\dot{\psi} = (q\sin\phi + r\cos\phi)\sec\theta$$

State Vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ \hline x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \\ x \\ y \\ z \\ \hline p \\ q \\ r \\ \phi \\ \theta \\ \psi \end{bmatrix}$$

23

Reorder the State Vector

Intermingled Axes

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \hline x_4 \\ x_5 \\ x_6 \\ \hline x_7 \\ x_8 \\ x_9 \\ \hline x_{10} \\ x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \\ \hline x \\ y \\ z \\ \hline p \\ q \\ r \\ \hline \phi \\ \theta \\ \psi \end{bmatrix}$$

First six elements of the state are longitudinal variables

Second six elements of the state are lateral-directional variables

Partitioned Axes

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ \hline x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \end{bmatrix}_{new} = \begin{bmatrix} \mathbf{x}_{Lon} \\ \hline \mathbf{x}_{Lat-Dir} \end{bmatrix} = \begin{bmatrix} u \\ w \\ x \\ z \\ q \\ \theta \\ \hline v \\ y \\ p \\ r \\ \phi \\ \psi \end{bmatrix}$$

24

Longitudinal Equations of Motion

Dynamics of velocity, position, angular rate, and angle primarily <u>in</u> the vertical plane

$$\dot{u} = X/m - g\sin\theta + rv - qw \triangleq \dot{x}_1 = f_1$$
$$\dot{w} = Z/m + g\cos\phi\cos\theta + qu - pv \triangleq \dot{x}_2 = f_2$$

$$\dot{x}_I = (\cos\theta\cos\psi)u + (-\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi)v + (\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi)w \triangleq \dot{x}_3 = f_3$$
$$\dot{z}_I = (-\sin\theta)u + (\sin\phi\cos\theta)v + (\cos\phi\cos\theta)w \triangleq \dot{x}_4 = f_4$$

$$\dot{q} = \left[M - (I_{xx} - I_{zz})pr - I_{xz}(p^2 - r^2)\right] \div I_{yy} \triangleq \dot{x}_5 = f_5$$
$$\dot{\theta} = q\cos\phi - r\sin\phi \triangleq \dot{x}_6 = f_6$$

$$\dot{\mathbf{x}}_{Lon}(t) = \mathbf{f}[\mathbf{x}_{Lon}(t), \mathbf{u}_{Lon}(t), \mathbf{w}_{Lon}(t)]$$

25

Lateral-Directional Equations of Motion

Dynamics of velocity, position, angular rate, and angle primarily <u>out of</u> the vertical plane

$$\dot{v} = Y/m + g\sin\phi\cos\theta - ru + pw \triangleq \dot{x}_7 = f_7$$
$$\dot{y}_I = (\cos\theta\sin\psi)u + (\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi)v + (-\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi)w \triangleq \dot{x}_8 = f_8$$

$$\dot{p} = \left(I_{zz}L + I_{xz}N - \left\{I_{xz}(I_{yy} - I_{xx} - I_{zz})p + \left[I_{xz}^2 + I_{zz}(I_{zz} - I_{yy})\right]r\right\}q\right) \div (I_{xx}I_{zz} - I_{xz}^2) \triangleq \dot{x}_9 = f_9$$
$$\dot{r} = \left(I_{xz}L + I_{xx}N - \left\{I_{xz}(I_{yy} - I_{xx} - I_{zz})r + \left[I_{xz}^2 + I_{xx}(I_{xx} - I_{yy})\right]p\right\}q\right) \div (I_{xx}I_{zz} - I_{xz}^2) \triangleq \dot{x}_{10} = f_{10}$$

$$\dot{\phi} = p + (q\sin\phi + r\cos\phi)\tan\theta \triangleq \dot{x}_{11} = f_{11}$$
$$\dot{\psi} = (q\sin\phi + r\cos\phi)\sec\theta \triangleq \dot{x}_{12} = f_{12}$$

$$\dot{\mathbf{x}}_{LD}(t) = \mathbf{f}[\mathbf{x}_{LD}(t), \mathbf{u}_{LD}(t), \mathbf{w}_{LD}(t)]$$

26

Sensitivity to Small Motions

(12 x 12) stability matrix for the entire system

$$\mathbf{F}(t) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{12}} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_{12}}{\partial x_1} & \cdots & \frac{\partial f_{12}}{\partial x_{12}} \end{bmatrix}$$

Four (6 x 6) blocks distinguish longitudinal and lateral-directional effects

Effects of longitudinal perturbations on longitudinal motion

Effects of lateral-directional perturbations on longitudinal motion

$$\mathbf{F} = \left[\begin{array}{c|c} \mathbf{F}_{Lon} & \mathbf{F}_{Lat-Dir}^{Lon} \\ \hline \mathbf{F}_{Lon}^{Lat-Dir} & \mathbf{F}_{Lat-Dir} \end{array}\right]$$

Effects of longitudinal perturbations on lateral-directional motion

Effects of lateral-directional perturbations on lateral-directional motion

27

Sensitivity to Small Control Inputs

(12 x 6) control matrix for the entire system

$$\mathbf{G}(t) = \left[\begin{array}{ccc|ccc} \partial f_1/\partial \delta E & \partial f_1/\partial \delta T & \partial f_1/\partial \delta F & \partial f_1/\partial \delta A & \partial f_1/\partial \delta R & \partial f_1/\partial \delta SF \\ \partial f_2/\partial \delta E & \partial f_2/\partial \delta T & \partial f_2/\partial \delta F & \partial f_2/\partial \delta A & \partial f_2/\partial \delta R & \partial f_2/\partial \delta SF \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \partial f_{12}/\partial \delta E & \partial f_{12}/\partial \delta T & \partial f_{12}/\partial \delta F & \partial f_{12}/\partial \delta A & \partial f_{12}/\partial \delta R & \partial f_{12}/\partial \delta SF \end{array}\right]$$

Four (6 x 3) blocks distinguish longitudinal and lateral-directional control effects

Effects of longitudinal controls on longitudinal motion

Effects of lateral-directional controls on longitudinal motion

$$\mathbf{G} = \left[\begin{array}{c|c} \mathbf{G}_{Lon} & \mathbf{G}_{Lat-Dir}^{Lon} \\ \hline \mathbf{G}_{Lon}^{Lat-Dir} & \mathbf{G}_{Lat-Dir} \end{array}\right]$$

Effects of longitudinal controls on lateral-directional motion

Effects of lateral-directional controls on lateral-directional motion

28

Sensitivity to Small Disturbance Inputs

Disturbance input vector and perturbation

$$\mathbf{w}(t) = \begin{bmatrix} u_w(t) \\ w_w(t) \\ q_w(t) \\ \hline v_w(t) \\ p_w(t) \\ r_w(t) \end{bmatrix} \begin{array}{c} \textit{Axial wind, m / s} \\ \textit{Normal wind, m / s} \\ \textit{Pitching wind shear, deg / s or rad / s} \\ \hline \textit{Lateral wind, m / s} \\ \textit{Rolling wind shear, deg / s or rad / s} \\ \textit{Yawing wind shear, deg / s or rad / s} \end{array}$$

$$\Delta\mathbf{w}(t) = \begin{bmatrix} \Delta u_w(t) \\ \Delta w_w(t) \\ \Delta q_w(t) \\ \hline \Delta v_w(t) \\ \Delta p_w(t) \\ \Delta r_w(t) \end{bmatrix}$$

Four (6 x 3) blocks **distinguish** longitudinal and lateral-directional effects

Effects of longitudinal disturbances on longitudinal motion

Effects of lateral-directional disturbances on longitudinal motion

$$\mathbf{L} = \left[\begin{array}{c|c} \mathbf{L}_{Lon} & \mathbf{L}_{Lat-Dir}^{Lon} \\ \hline \mathbf{L}_{Lon}^{Lat-Dir} & \mathbf{L}_{Lat-Dir} \end{array} \right]$$

Effects of longitudinal disturbances on lateral-directional motion

Effects of lateral-directional disturbances on lateral-directional motion

Decoupling Approximation for Small Perturbations from Steady, Level Flight

Restrict the Nominal Flight Path to the Vertical Plane

Nominal lateral-directional motions are zero

$$\dot{\mathbf{x}}_{Lat-Dir_N} = \mathbf{0}$$
$$\mathbf{x}_{Lat-Dir_N} = \mathbf{0}$$

Nominal longitudinal equations reduce to

$$\dot{\mathbf{x}}_{Lon}(t) = \mathbf{f}_{Lon}\left[\mathbf{x}_{Lon}(t), \mathbf{u}_{Lon}(t)\right]$$

$$\begin{aligned}
\dot{u}_N &= X/m - g\sin\theta_N - q_N w_N \\
\dot{w}_N &= Z/m + g\cos\theta_N + q_N u_N \\
\dot{x}_{I_N} &= (\cos\theta_N)u_N + (\sin\theta_N)w_N \\
\dot{z}_{I_N} &= (-\sin\theta_N)u_N + (\cos\theta_N)w_N \\
\dot{q}_N &= \frac{M}{I_{yy}} \\
\dot{\theta}_N &= q_N
\end{aligned}$$

Nominal State Vector

$$\left[\begin{array}{c} \mathbf{x}_{Lon}(t) \\ \hline \mathbf{x}_{Lat-Dir}(t) \end{array}\right]_{Nominal} = \left[\begin{array}{c} u_N(t) \\ w_N(t) \\ x_N(t) \\ z_N(t) \\ q_N(t) \\ \theta_N(t) \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

31

Restrict the Nominal Flight Path to the Vertical Plane

However, lateral-directional perturbations may not be zero

$$\Delta\dot{\mathbf{x}}_{Lat-Dir} \neq \mathbf{0}$$
$$\Delta\mathbf{x}_{Lat-Dir} \neq \mathbf{0}$$

$$\left[\begin{array}{c} \Delta\dot{\mathbf{x}}_{Lon}(t) \\ \Delta\dot{\mathbf{x}}_{Lat-Dir}(t) \end{array}\right] = \left[\begin{array}{c|c} \mathbf{F}_{Lon}(t) & \sim\mathbf{0} \\ \hline \sim\mathbf{0} & \mathbf{F}_{Lat-Dir}(t) \end{array}\right]\left[\begin{array}{c} \Delta\mathbf{x}_{Lon}(t) \\ \Delta\mathbf{x}_{Lat-Dir}(t) \end{array}\right] + \left[\begin{array}{c|c} \mathbf{G}_{Lon}(t) & \sim\mathbf{0} \\ \hline \sim\mathbf{0} & \mathbf{G}_{Lat-Dir}(t) \end{array}\right]\left[\begin{array}{c} \Delta\mathbf{u}_{Lon}(t) \\ \Delta\mathbf{u}_{Lat-Dir}(t) \end{array}\right]$$

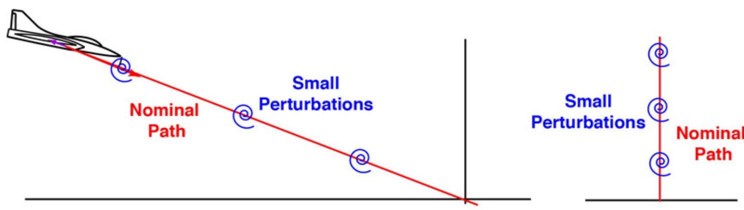
[See Section 4.1, *Flight Dynamics*, for F and G definitions]

Perturbation State Vector

$$\left[\begin{array}{c} \Delta\mathbf{x}_{Lon}(t) \\ \hline \Delta\mathbf{x}_{Lat-Dir}(t) \end{array}\right] = \left[\begin{array}{c} \Delta u(t) \\ \Delta w(t) \\ \Delta x(t) \\ \Delta z(t) \\ \Delta q(t) \\ \Delta\theta(t) \\ \hline \Delta v(t) \\ \Delta y(t) \\ \Delta p(t) \\ \Delta r(t) \\ \Delta\phi(t) \\ \Delta\psi(t) \end{array}\right]$$

32

Nominal (Reference) Motion Plus Perturbation



$$\dot{\mathbf{x}}_N(t) = \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t)]$$

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}(t)\Delta\mathbf{x}(t) + \mathbf{G}(t)\Delta\mathbf{u}(t)$$

33

Restrict the <u>Nominal</u> Flight Path to Steady, Level Flight

Longitudinal ODEs replaced by Algebraic Equations for Equilibrium (Trim)

$$
\begin{aligned}
0 &= X/m - g\sin\theta_N - q_N w_N \\
0 &= Z/m + g\cos\theta_N + q_N u_N \\
V_N &= (\cos\theta_N)u_N + (\sin\theta_N)w_N \\
0 &= (-\sin\theta_N)u_N + (\cos\theta_N)w_N \\
0 &= \frac{M}{I_{yy}} \\
0 &= q_N
\end{aligned}
$$

Trimmed State Vector (except x_N) is constant

$$
\begin{bmatrix} u \\ w \\ x \\ z \\ q \\ \theta \end{bmatrix}_{Trim} = \begin{bmatrix} u_{Trim} \\ w_{Trim} \\ V_N(t-t_0) \\ z_N \\ 0 \\ \theta_{Trim} \end{bmatrix}
$$

Perturbation Model: F and G are constant

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}(t)\Delta\mathbf{x}(t) + \mathbf{G}(t)\Delta\mathbf{u}(t) = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t)$$

Longitudinal and lateral-directional perturbations are essentially decoupled

$$
\begin{bmatrix} \Delta\dot{\mathbf{x}}_{Lon}(t) \\ \Delta\dot{\mathbf{x}}_{Lat-Dir}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{Lon} & \sim\mathbf{0} \\ \sim\mathbf{0} & \mathbf{F}_{Lat-Dir} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}_{Lon}(t) \\ \Delta\mathbf{x}_{Lat-Dir}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{G}_{Lon} & \sim\mathbf{0} \\ \sim\mathbf{0} & \mathbf{G}_{Lat-Dir} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u}_{Lon}(t) \\ \Delta\mathbf{u}_{Lat-Dir}(t) \end{bmatrix}
$$

34

Small Longitudinal and Lateral-Directional Perturbation Effects

- Assume the airplane is symmetric and its nominal path is steady, level flight
  - Small longitudinal and lateral-directional perturbations are approximately uncoupled from each other
  - (12 x 12) system is
    - block diagonal
    - constant, i.e., linear, time-invariant (LTI)
    - DECOUPLED into two separate (6 x 6) systems

$$\mathbf{F} = \left[\begin{array}{c|c} \mathbf{F}_{Lon} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{F}_{Lat-Dir} \end{array}\right] \quad \mathbf{G} = \left[\begin{array}{c|c} \mathbf{G}_{Lon} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{G}_{Lat-Dir} \end{array}\right] \quad \mathbf{L} = \left[\begin{array}{c|c} \mathbf{L}_{Lon} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{L}_{Lat-Dir} \end{array}\right]$$

35

(6 x 6) LTI Longitudinal Perturbation Model

Dynamic Equation

$$\Delta\dot{\mathbf{x}}_{Lon}(t) = \mathbf{F}_{Lon}\Delta\mathbf{x}_{Lon}(t) + \mathbf{G}_{Lon}\Delta\mathbf{u}_{Lon}(t) + \mathbf{L}_{Lon}\Delta\mathbf{w}_{Lon}(t)$$

State Vector

$$\Delta\mathbf{x}_{Lon} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \\ \Delta x_5 \\ \Delta x_6 \end{bmatrix}_{Lon} = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta x \\ \Delta z \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

Control Vector

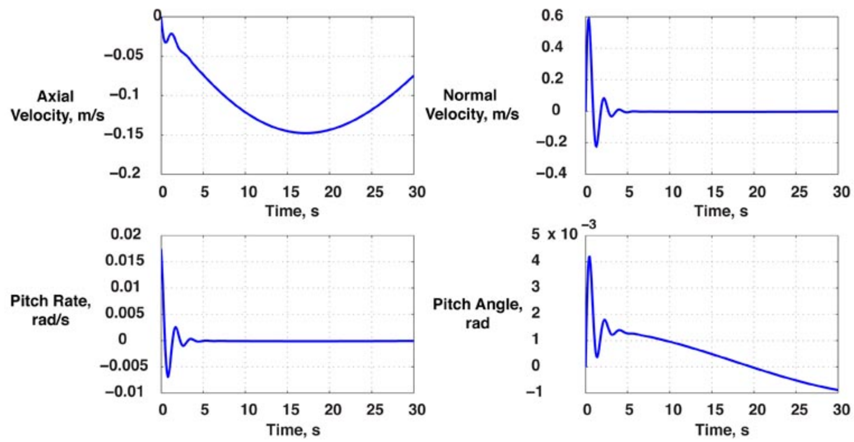
$$\Delta\mathbf{u}_{Lon} = \begin{bmatrix} \Delta\delta T \\ \Delta\delta E \\ \Delta\delta F \end{bmatrix}$$

Disturbance Vector

$$\Delta\mathbf{w}_{Lon} = \begin{bmatrix} \Delta u_{wind} \\ \Delta w_{wind} \\ \Delta q_{wind} \end{bmatrix}$$

36

LTI Longitudinal Response to Initial Pitch Rate



37

(6 x 6) LTI Lateral-Directional Perturbation Model

Dynamic Equation

$$\Delta\dot{\mathbf{x}}_{Lat-Dir}(t) = \mathbf{F}_{Lat-Dir}\Delta\mathbf{x}_{Lat-Dir}(t) + \mathbf{G}_{Lat-Dir}\Delta\mathbf{u}_{Lat-Dir}(t) + \mathbf{L}_{Lat-Dir}\Delta\mathbf{w}_{Lat-Dir}(t)$$

State Vector

$$\Delta\mathbf{x}_{Lat-Dir} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \\ \Delta x_5 \\ \Delta x_6 \end{bmatrix}_{Lat-Dir} = \begin{bmatrix} \Delta v \\ \Delta y \\ \Delta p \\ \Delta r \\ \Delta \phi \\ \Delta \psi \end{bmatrix}$$

Control Vector

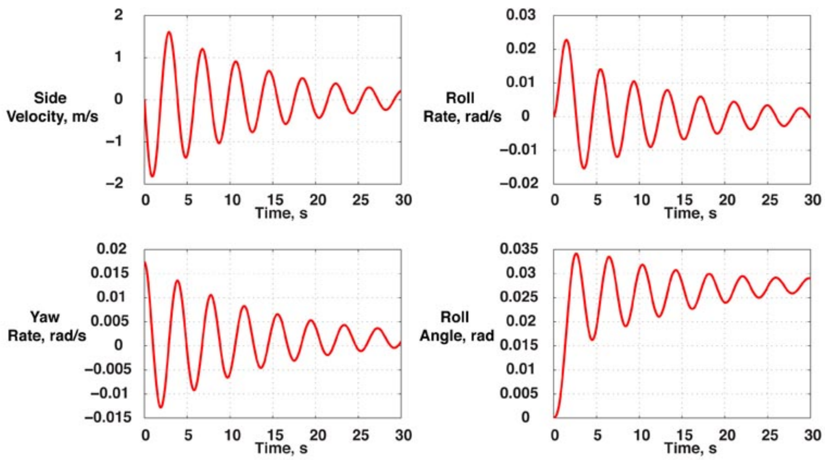
$$\Delta\mathbf{u}_{Lat-Dir} = \begin{bmatrix} \Delta\delta A \\ \Delta\delta R \\ \Delta\delta SF \end{bmatrix}$$

Disturbance Vector

$$\Delta\mathbf{w}_{Lon} = \begin{bmatrix} \Delta v_{wind} \\ \Delta p_{wind} \\ \Delta r_{wind} \end{bmatrix}$$

38

LTI Lateral-Directional Response to Initial Yaw Rate



39

Next Time: Longitudinal Dynamics

Reading:
Flight Dynamics
452-464, 482-486
Airplane Stability and Control
Chapter 7

Learning Objectives

- **6^{th}-order -> 4^{th}-order -> hybrid equations**
- **Dynamic stability derivatives**
- **Long-period (phugoid) mode**
- **Short-period mode**

40

Supplemental Material

How Do We Calculate the Partial Derivatives?

$$\mathbf{F}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t) \\ \mathbf{u}=\mathbf{u}_N(t) \\ \mathbf{w}=\mathbf{w}_N(t)}} \quad \mathbf{G}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t) \\ \mathbf{u}=\mathbf{u}_N(t) \\ \mathbf{w}=\mathbf{w}_N(t)}} \quad \mathbf{L}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{w}}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t) \\ \mathbf{u}=\mathbf{u}_N(t) \\ \mathbf{w}=\mathbf{w}_N(t)}}$$

- **Analytically**
  - **Symbolic evaluation of analytical models of F, G, and L**
- **Numerically**
  - **First central differences in f(x,u,w)**

Numerical Estimation of the Jacobian Matrix, F(*t*)

$$\frac{\partial f_1}{\partial x_1}(t) \approx \frac{\left. f_1\begin{bmatrix}(x_1+\Delta x_1)\\ x_2\\ \ldots\\ x_n\end{bmatrix} - f_1\begin{bmatrix}(x_1-\Delta x_1)\\ x_2\\ \ldots\\ x_n\end{bmatrix}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta x_1}; \quad \frac{\partial f_1}{\partial x_2}(t) \approx \frac{\left. f_1\begin{bmatrix}x_1\\ (x_2+\Delta x_2)\\ \ldots\\ x_n\end{bmatrix} - f_1\begin{bmatrix}x_1\\ (x_2-\Delta x_2)\\ \ldots\\ x_n\end{bmatrix}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta x_2}$$

$$\frac{\partial f_2}{\partial x_1}(t) \approx \frac{\left. f_2\begin{bmatrix}(x_1+\Delta x_1)\\ x_2\\ \ldots\\ x_n\end{bmatrix} - f_2\begin{bmatrix}(x_1-\Delta x_1)\\ x_2\\ \ldots\\ x_n\end{bmatrix}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta x_1}; \quad \frac{\partial f_2}{\partial x_2}(t) \approx \frac{\left. f_2\begin{bmatrix}x_1\\ (x_2+\Delta x_2)\\ \ldots\\ x_n\end{bmatrix} - f_2\begin{bmatrix}x_1\\ (x_2-\Delta x_2)\\ \ldots\\ x_n\end{bmatrix}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta x_2}$$

Continue for all *n* x *n* elements of F(*t*)

43

Numerical Estimation of the Jacobian Matrix, G(*t*)

$$\frac{\partial f_1}{\partial u_1}(t) \approx \frac{\left. f_1\begin{bmatrix}(u_1+\Delta u_1)\\ u_2\\ \ldots\\ u_m\end{bmatrix} - f_1\begin{bmatrix}(u_1-\Delta u_1)\\ u_2\\ \ldots\\ u_m\end{bmatrix}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta u_1}; \quad \frac{\partial f_1}{\partial u_2}(t) \approx \frac{\left. f_1\begin{bmatrix}u_1\\ (u_2+\Delta u_2)\\ \ldots\\ u_m\end{bmatrix} - f_1\begin{bmatrix}u_1\\ (u_2-\Delta u_2)\\ \ldots\\ u_m\end{bmatrix}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta u_2}$$

$$\frac{\partial f_2}{\partial u_1}(t) \approx \frac{\left. f_2\begin{bmatrix}(u_1+\Delta u_1)\\ u_2\\ \ldots\\ u_m\end{bmatrix} - f_2\begin{bmatrix}(u_1-\Delta u_1)\\ u_2\\ \ldots\\ u_m\end{bmatrix}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta u_1}; \quad \frac{\partial f_2}{\partial u_2}(t) \approx \frac{\left. f_2\begin{bmatrix}u_1\\ (u_2+\Delta u_2)\\ \ldots\\ u_m\end{bmatrix} - f_2\begin{bmatrix}u_1\\ (u_2-\Delta u_2)\\ \ldots\\ u_m\end{bmatrix}\right|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta u_2}$$

Continue for all *n* x *m* elements of G(*t*)

44

Numerical Estimation of the Jacobian Matrix, L(*t*)

$$\frac{\partial f_1}{\partial w_1}(t) \approx \frac{f_1\begin{bmatrix}(w_1+\Delta w_1)\\ w_2\\ \cdots\\ w_s\end{bmatrix} - f_1\begin{bmatrix}(w_1-\Delta w_1)\\ w_2\\ \cdots\\ w_s\end{bmatrix}}{2\Delta w_1}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}; \quad \frac{\partial f_1}{\partial w_2}(t) \approx \frac{f_1\begin{bmatrix}w_1\\ (w_2+\Delta w_2)\\ \cdots\\ w_s\end{bmatrix} - f_1\begin{bmatrix}w_1\\ (w_2-\Delta w_2)\\ \cdots\\ w_s\end{bmatrix}}{2\Delta w_2}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}$$

$$\frac{\partial f_2}{\partial w_1}(t) \approx \frac{f_2\begin{bmatrix}(w_1+\Delta w_1)\\ w_2\\ \cdots\\ w_s\end{bmatrix} - f_2\begin{bmatrix}(w_1-\Delta w_1)\\ w_2\\ \cdots\\ w_s\end{bmatrix}}{2\Delta w_1}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}; \quad \frac{\partial f_2}{\partial w_2}(t) \approx \frac{f_2\begin{bmatrix}w_1\\ (w_2+\Delta w_2)\\ \cdots\\ w_s\end{bmatrix} - f_2\begin{bmatrix}w_1\\ (w_2-\Delta w_2)\\ \cdots\\ w_s\end{bmatrix}}{2\Delta w_2}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}$$

Continue for all *n* x *s* elements of L(*t*)

45